

# STOCHASTIC AFFINE EVOLUTION EQUATIONS WITH MULTIPLICATIVE FRACTIONAL NOISE

B. MASLOWSKI AND J. ŠNUPÁRKOVÁ

**ABSTRACT.** *A stochastic affine evolution equation with bilinear noise term is studied where the driving process is a real-valued fractional Brownian motion. Stochastic integration is understood in the Skorokhod sense. Existence and uniqueness of weak solution is proved and some results on the large time dynamics are obtained.*

## 1. INTRODUCTION

In the paper the formula for stochastic evolution system generated by equation with bilinear stochastic term and affine drift term is studied. Existence and uniqueness of solutions is proved and the relation between weak and “mild” form of solutions is investigated. Some peculiarities of large time behaviour are also demonstrated. The results obtained for the equation in the general infinite-dimensional form are applied to linear stochastic PDE of second order.

Stochastic differential equations in Hilbert spaces with multiplicative white noise have been studied in numerous papers, e.g. G. Da Prato, M. Iannelli, L. Tubaro [5], [4], F. Flandoli [9], and in Chapter 6 of the monograph by G. Da Prato and J. Zabczyk [6]. The solution to such equations may be viewed as a generalization of the geometric Brownian motion, which has a wide range of applications. In all these cases the driving process is the Brownian motion. Later, S. Bonaccorsi in [3] studied mild solutions of equations with additional nonlinear terms in the drift and diffusion parts, defined by means of the stochastic evolution system induced by the bilinear equation.

Analogous results have been obtained for bilinear evolution equations of the same type with fractional Gaussian noise by T. Duncan, B. Maslowski, B. Pasik-Duncan [7] (for  $H > 1/2$ , where  $H$  denotes the Hurst parameter of the driving fractional Brownian motion) and J. Šnupárková [17] (for  $H < 1/2$ ). In these papers the stochastic integral is understood in the Skorokhod sense, i.e. as the adjoint operator to the Malliavin derivative. On the other hand, semilinear evolution equations with bilinear Stratonovich noise have been studied in [10]. It was shown that the equation defines a random dynamical system (which is not true in the case of Skorokhod integration) and the long-time behaviour was dealt with.

The present paper is organized as follows. In Section 2 the notion of Skorokhod integral with respect to fractional Brownian motion and its basic properties are recalled. Section 3 is devoted to an extension of a result from [7] on the existence of a weak solution to the bilinear equation. In Section 4 the existence and uniqueness of the mild solution is proved (Theorem 4.1). If the perturbation  $F$  does not depend on the solution process, the mild solution of the corresponding affine equation is the weak one (Theorem 4.6). Unlike in the

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case of the standard Brownian motion, this is no longer true if the equation is semilinear (the perturbation  $F$  depends on the solution) as shown by a simple counterexample. Note that it is not completely clear how to define the candidate on the “random evolution system”, as explained in Example 4.4. Following the ideas from [3] this system is used to define the mild solution of an equation with additional nonlinearity in the drift part for  $H > 1/2$ . While the mild formulation implies a weak one in the Wiener case it need not be true in the case of fractional Brownian motion unless the perturbation is independent of the solution process. It is shown that this “mild” solution satisfies certain different equation in the weak sense.

Some large-time behaviour results are also proved. Sections 5 and 6 are devoted to the proof of uniqueness of solutions to the affine equation. This problem is nontrivial because the Gronwall lemma is not applicable as in the case of standard Brownian motion. Instead, we prove uniqueness of the mild solution to the bilinear equation inductively, showing uniqueness of the coefficients in the Wiener chaos expansions, and using this result we prove the uniqueness of weak solutions to the affine equation.

## 2. PRELIMINARIES

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space. A stochastic process  $B^H = \{B_t^H, t \in [0, T]\}$  is a **fractional Brownian motion** with Hurst parameter  $H \in (0, 1)$  if it is a real-valued centered Gaussian process with the covariance function given by

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H}), \quad s, t \geq 0.$$

In what follows Hurst parameter  $H > 1/2$  is assumed.

Define the linear operator  $\mathcal{K}_H^* : \mathcal{E} \rightarrow L^2([0, T])$  as

$$(\mathcal{K}_H^* f)(t) = C_H \Gamma\left(H - \frac{1}{2}\right) t^{\frac{1}{2}-H} \left(I_{T-}^{H-\frac{1}{2}} f_{H-\frac{1}{2}}\right)(t),$$

where  $f_{H-\frac{1}{2}}(t) = t^{H-\frac{1}{2}} f(t)$ ,  $t \in [0, T]$ ,  $f \in \mathcal{E}$  is a step function of the form

$$\varphi = \sum_{k=0}^{N-1} a_k I_{(t_k, t_{k+1}]} ,$$

for some  $N \in \mathbb{N}$ ,  $0 = t_0 < t_1 < \dots < t_N = T$ ,  $a_k \in \mathbb{R}$ ,  $k = 0, \dots, N$ ,  $I_{T-}^{H-\frac{1}{2}}$  is a Riemann–Liouville fractional right-sided integral defined as

$$(I_{T-}^{H-\frac{1}{2}} f)(t) = \frac{1}{\Gamma(H - \frac{1}{2})} \int_t^T \frac{f(s)}{(s - t)^{\frac{3}{2}-H}} ds \quad \text{for a.e. } t \in [0, T],$$

and

$$C_H = \sqrt{\frac{H(2H - 1)}{\Gamma(2 - 2H, H - \frac{1}{2})}}.$$

Using the operator  $\mathcal{K}_H^*$  define the scalar product on  $\mathcal{E}$  as

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := \langle \mathcal{K}_H^*(\varphi), \mathcal{K}_H^*(\psi) \rangle_{L^2([0, T])}, \quad \varphi, \psi \in \mathcal{E}.$$

Denote by  $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$  the Hilbert space obtained as the completion of  $\mathcal{E}$  with respect to the scalar product  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  and let  $\|\cdot\|_{\mathcal{H}}$  be the norm induced by  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ .

For  $\varphi \in \mathcal{E}$  define the stochastic integral with respect to the fractional Brownian motion

$$I(\varphi) \equiv \int_0^T \varphi(s) dB_s^H := \sum_{k=0}^{N-1} a_k (B^H(t_{k+1}) - B^H(t_k)).$$

Since

$$\mathbb{E} \left[ \int_0^T \varphi(s) dB_s^H \int_0^T \psi(s) dB_s^H \right] = \langle \varphi, \psi \rangle_{\mathcal{H}}, \quad \varphi, \psi \in \mathcal{E},$$

(see [1]), the integral can be uniquely extended to  $\mathcal{H}$  (the standard notation  $I(\varphi) = B^H(\varphi) = \int_0^T \varphi(r) dB_r^H$  is also used) and the operator  $\mathcal{K}_H^*$  provides an isometry between spaces  $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$  and  $L^2(\Omega)$ .

Let  $\mathcal{S}$  be a set of smooth cylindrical random variables of the form

$$F = f(B^H(\varphi_1), \dots, B^H(\varphi_n)), \quad (2.1)$$

where  $n \geq 1$ ,  $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$  ( $f$  and all its partial derivatives are bounded) and  $\varphi_i \in \mathcal{H}$ ,  $i = 1, \dots, n$ . The **derivative operator (Malliavin derivative)** of a smooth cylindrical random variable  $F$  of the form (2.1) is an  $\mathcal{H}$ -valued random variable

$$D^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (B^H(\varphi_1), \dots, B^H(\varphi_n)) \varphi_i.$$

The derivative operator  $D^H$  is closable from  $L^p(\Omega)$  into  $L^p(\Omega; \mathcal{H})$  for any  $p \in [1, +\infty)$ . Let  $\mathbb{D}_H^{1,p}$  be the Sobolev space obtained as a closure of  $\mathcal{S}$  with respect to the norm

$$\|F\|_{1,p} := \left( \mathbb{E} [|F|^p] + \mathbb{E} [\|D^H F\|_{\mathcal{H}}^p] \right)^{1/p}$$

for any  $p \in [1, +\infty)$ . Similarly, given a Hilbert space  $\tilde{V} \subset \mathcal{H}$ , set  $\mathbb{D}_H^{1,p}(\tilde{V})$  for the corresponding Sobolev space of  $\tilde{V}$ -valued random variables.

**Definition 2.1.** The **divergence operator (Skorokhod integral)**  $\delta_H : \text{Dom } \delta_H \rightarrow L^2(\Omega)$  is defined as the adjoint operator of the derivative operator  $D^H : L^2(\Omega) \rightarrow L^2(\Omega; \mathcal{H})$ , i.e. for any  $u \in \text{Dom } \delta_H$  the duality relationship

$$\mathbb{E} [F \delta_H(u)] = \mathbb{E} [\langle D^H F, u \rangle_{\mathcal{H}}]$$

holds for any  $F \in \mathbb{D}_H^{1,2}$ .

A random variable  $u \in L^2(\Omega; \mathcal{H})$  belongs to the domain  $\text{Dom } \delta_H$  if there exists a constant  $c_u < +\infty$  depending only on  $u$  such that

$$|\mathbb{E} [\langle D^H F, u \rangle_{\mathcal{H}}]| \leq c_u \|F\|_{L^2(\Omega)}$$

for any  $F \in \mathcal{S}$ .

The useful facts listed below can be found e.g. in [14]. Let  $|\mathcal{H}| \subset \mathcal{H}$  be a linear space of measurable functions  $\varphi$  on  $[0, T]$  such that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi(r)| |\varphi(s)| |r - s|^{2H-2} dr ds < +\infty,$$

where  $\alpha_H = H(2H-1)$ . Then  $\mathcal{E}$  is dense in  $|\mathcal{H}|$  and  $(|\mathcal{H}|, \|\cdot\|_{|\mathcal{H}|})$  is a Banach space. Moreover,

$$L^2([0, T]) \subset L^{1/H}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H},$$

thus there exists a constant  $K_e < +\infty$  such that

$$\|\mathcal{K}_H^*(\varphi)\|_{L^2([0,T])} = \|\varphi\|_{\mathcal{H}} \leq K_e \|\varphi\|_{L^2([0,T])} \quad (2.2)$$

for any  $\varphi \in \mathcal{H}$ . Note that

$$\mathbb{D}_H^{1,2}(|\mathcal{H}|) \subset \mathbb{D}_H^{1,2}(\mathcal{H}) \subset \text{Dom } \delta_H \quad (2.3)$$

and for some constant  $\tilde{C}_{H,2} < +\infty$

$$\mathbb{E} [\delta_H^2(u)] \leq \tilde{C}_{H,2} \left( \mathbb{E} [\|u\|_{|\mathcal{H}|}^2] + \mathbb{E} [\|D^H u\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^2] \right), \quad u \in \mathbb{D}_H^{1,2}(|\mathcal{H}|),$$

holds, where  $\mathbb{D}_H^{1,p}(|\mathcal{H}|)$  ( $p \in (1, +\infty)$ ) contains processes  $u \in \mathbb{D}_H^{1,p}(\mathcal{H})$  such that  $u \in |\mathcal{H}|$ ,  $D^H u \in |\mathcal{H}| \otimes |\mathcal{H}|$   $\mathbb{P}$ -a.s. and

$$\mathbb{E} [\|u\|_{|\mathcal{H}|}^p] + \mathbb{E} [\|D^H u\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^p] < +\infty.$$

The normed linear space  $(|\mathcal{H}| \otimes |\mathcal{H}|, \|\cdot\|_{|\mathcal{H}| \otimes |\mathcal{H}|})$  is defined in a similar way as  $(|\mathcal{H}|, \|\cdot\|_{|\mathcal{H}|})$  (for a precise definition see e.g. [14]). Hence, for some constant  $C_{H,2} < +\infty$

$$\mathbb{E} [\delta_H^2(u)] \leq C_{H,2} \left( \mathbb{E} [\|u\|_{L^{1/H}([0,T])}^2] + \mathbb{E} [\|D^H u\|_{L^{1/H}([0,T]^2)}^2] \right), \quad u \in \mathbb{D}_H^{1,2}(|\mathcal{H}|). \quad (2.4)$$

Since the process  $B^H$  has an integral representation (see e.g. [14])

$$B_t^H = \int_0^t (\mathcal{K}_H^* I_{(0,t]})(s) dW_s, \quad t \geq 0, \quad (2.5)$$

where  $W = \{W_t, t \geq 0\}$  is a Wiener process on  $(\Omega, \mathcal{F}, \mathbb{P})$ , similar relations are valid for derivatives and divergence operators, i.e.

(i) for any  $F \in \mathbb{D}_W^{1,2}$

$$\mathcal{K}_H^*(D^H F) = D^W F,$$

where  $D^W$  denotes the derivative operator with respect to  $W$  and  $\mathbb{D}_W^{1,2}$  the corresponding Sobolev space,

(ii)  $\text{Dom } \delta_W = \mathcal{K}_H^*(\text{Dom } \delta_H)$  and

$$\delta_H(u) = \delta_W(\mathcal{K}_H^* u) \quad (2.6)$$

for any  $u \in \text{Dom } \delta_H$ , where  $\delta_W$  denotes the divergence operator with respect to  $W$ .

*Remark 2.2.* The construction (and the properties) of Malliavin derivative and Skorokhod integral for Hilbert space-valued random variables are completely analogous.

### 3. RANDOM EVOLUTION SYSTEM

In this short overview section, a result from [7] is slightly extended to obtain a random two-parameter evolution system representing the solution to the equation

$$\begin{aligned} dY_t &= AY_t dt + BY_t dB_t^H, \quad t > s, \\ Y_s &= x, \end{aligned} \quad (3.1)$$

in a separable Hilbert space  $V$  on a finite interval  $[0, T]$  with general initial time  $s \in [0, T]$  and deterministic initial value  $x \in V$ . The driving process  $\{B_t^H, t \geq 0\}$  is a one-dimensional fractional Brownian motion with Hurst parameter  $H > 1/2$  on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and the stochastic integral is understood in the Skorokhod sense (see [1] for more details).

The linear operators  $A$  and  $B$  on  $V$  satisfy

- (A1) the operator  $A$  is closed and densely defined with the domain  $D := \text{Dom}(A)$ ,  
 (A2) the resolvent set contains all  $\lambda \in \mathbb{C}$  such that  $\text{Re}(\lambda) \geq \omega$  for some fixed  $\omega \in \mathbb{R}$  and for some constant  $M > 0$  the resolvent  $R(\lambda, A)$  satisfies

$$\|R(\lambda, A)\|_{\mathcal{L}(V)} \leq \frac{M}{|\lambda - \omega| + 1}$$

for all  $\lambda \in \mathbb{C}$ ,  $\text{Re}(\lambda) \geq \omega$ , where  $\mathcal{L}(V)$  stands for a space of all linear bounded operators on  $V$ ,

- (B2) the operator  $B$  is closed, densely defined and generates a strongly continuous group  $\{S_B(u), u \in \mathbb{R}\}$  on  $V$ .

The conditions (A1) and (A2) imply that the operator  $A$  generates an analytic semigroup  $\{S_A(t), 0 \leq t \leq T\}$  on  $V$ . The condition (B2) ensures the existence of constants  $M_B \geq 1$ ,  $\omega_B \geq 0$  such that the inequality

$$\|S_B(u)\|_{\mathcal{L}(V)} \leq M_B \exp\{\omega_B |u|\} \quad (3.2)$$

holds for each  $u \in \mathbb{R}$ .

For simplicity assume that  $\omega < 0$  (cf.(A2)). Note that since the operator  $-A$  is sectorial, the fractional powers  $(-A)^\alpha$  for  $\alpha \in (0, 1]$  are well-defined (see e.g. [15]), so the following condition makes sense. Suppose that

- (B3)  $B^2$  is closed and

$$\text{Dom}(B^2) \supset \text{Dom}((-A)^\alpha) \quad (3.3)$$

for some  $\alpha \in (0, 1)$ .

Define the operators  $\bar{A}(t) : D \rightarrow V$  as

$$\bar{A}(t) = A - Ht^{2H-1}B^2$$

for any  $t \in [0, T]$ .

**Lemma 3.1.** *Under the assumptions (A1), (A2), (B2) and (B3) the system  $\{\bar{A}(t), t \in [0, T]\}$  generates a strongly continuous evolution system  $\{U(t, s), 0 \leq s \leq t \leq T\}$  on  $V$ .*

*Proof.* See [7]. □

The system  $\{U(t, s), 0 \leq s \leq t \leq T\}$  satisfies

$$\begin{aligned} \text{Im}(U(t, s)) &\subset D, \\ \|U(t, s)\|_{\mathcal{L}(V)} &\leq C_U, \\ \left\| \frac{\partial}{\partial t} U(t, s) \right\|_{\mathcal{L}(V)} &= \|\bar{A}(t)U(t, s)\|_{\mathcal{L}(V)} \leq \frac{C_U}{t-s}, \\ \|\bar{A}(t)U(t, s)(\bar{A}(s) - \bar{\omega}I)^{-1}\|_{\mathcal{L}(V)} &\leq C_U \end{aligned} \quad (3.4)$$

for some constants  $C_U > 0$ ,  $\bar{\omega} \in \mathbb{R}$  and any  $0 \leq s < t \leq T$  (see e.g. [18], Theorem 5.2.1).

*Remark 3.2.* Instead of (A2) and (B3) we may assume directly that  $\{\bar{A}(t), t \in [0, T]\}$  generates a strongly continuous evolution system  $\{U(t, s), 0 \leq s \leq t \leq T\}$  on  $V$ . Nevertheless, the condition (A2) can be useful in applications to stochastic partial differential equations (as shown in [7]).

Let  $A^*$  denote the adjoint operator to the operator  $A$ . Let  $\text{Dom}(A^*) = D^*$  be the domain of  $A^*$  and suppose that

- (B1)  $D^* \subset \text{Dom}((B^*)^2)$ .

**Definition 3.3.** A  $(\mathcal{B}([s, T]) \otimes \mathcal{F})$ -measurable stochastic process  $\{Y_t, t \in [s, T]\}$  is said to be a **weak solution** to the equation (3.1) if for any  $y \in D^*$

$$\langle Y_t, y \rangle_V = \langle x, y \rangle_V + \int_s^t \langle Y_r, A^* y \rangle_V dr + \int_s^t \langle Y_r, B^* y \rangle_V dB_r^H \quad \mathbb{P}\text{-a.s.}$$

for all  $t \in [s, T]$ , where the integrals have to be well-defined.

**Theorem 3.4.** Let

(AB) the operators  $A$  and  $\{S_B(u), u \in \mathbb{R}\}$  commute on the domain  $D$ , i.e.

$$S_B(u)Ay = AS_B(u)y$$

for any  $u \in \mathbb{R}$  and  $y \in D$ .

The process  $\{U_Y(t, s)x, s \leq t \leq T\}$  defined as

$$U_Y(t, s)x = S_B(B_t^H - B_s^H)U(t - s, 0)x, \quad s \leq t \leq T, \quad (3.5)$$

is a weak solution to the equation (3.1) for any fixed  $x \in V$  and  $s \in [0, T]$  under the assumptions (A1), (A2) and (B1), (B2), (B3).

*Proof.* The proof is completely analogous to the proof of Theorem 2.3 in [7].  $\square$

*Remark 3.5.* The system  $\{U_Y(t, s), 0 \leq s \leq t \leq T\}$  is not a random continuous evolution system because it does not possess the standard composition property.

#### 4. PERTURBED EQUATION

In this section the equation with a perturbation in the drift part is studied.

Let  $H > 1/2$  and  $\{U_Y(t, s), 0 \leq s \leq t \leq T\}$  be the system of operators defined as

$$U_Y(t, s)x := S_B(B_t^H - B_s^H)U(t - s, 0)x, \quad x \in V,$$

where  $\{U(t, s), 0 \leq s \leq t \leq T\}$  is a strongly continuous evolution system associated with operators  $\{A - Ht^{2H-1}B^2, t \in [0, T]\}$  and  $\{S_B(u), u \in \mathbb{R}\}$  is a strongly continuous group associated with operator  $B$  satisfying conditions from Theorem 3.4. Note that in the previous section it has been shown that for any fixed  $s \in [0, T]$  the process  $\{U_Y(t, s)x, s \leq t \leq T\}$  is a weak solution to the equation

$$\begin{aligned} dY_t &= AY_t dt + BY_t dB_t^H, \quad t > s, \\ Y_s &= x \in V. \end{aligned} \quad (4.1)$$

**Theorem 4.1.** Let  $F : [0, T] \times V \rightarrow V$  be a measurable function satisfying

(i)<sub>F</sub> there exists a function  $\bar{L} \in L^1([0, T])$  such that

$$\|F(t, x) - F(t, y)\|_V \leq \bar{L}(t)\|x - y\|_V, \quad x, y \in V, \quad t \in [0, T],$$

(ii)<sub>F</sub> for some function  $\bar{K} \in L^1([0, T])$

$$\|F(t, 0)\|_V \leq \bar{K}(t), \quad t \in [0, T].$$

Then the equation

$$y(t) = U_Y(t, 0)x + \int_0^t U_Y(t, r)F(r, y(r))dr \quad (4.2)$$

has a unique solution in the space  $\mathcal{C}([0, T]; V)$  for a.e.  $\omega \in \Omega$  and any initial value  $x \in V$ .

*Proof.* Fix  $x \in V$  and show that the mapping

$$(\mathcal{K}(y))(t) = U_Y(t, 0)x + \int_0^t U_Y(t, r)F(r, y(r))dr$$

is continuous from  $\mathcal{C}([0, T]; V)$  into  $\mathcal{C}([0, T]; V)$  and that  $\mathcal{K}$  is a contraction mapping. Take  $y \in \mathcal{C}([0, T]; V)$  and  $t, s \in [0, T]$ . Then

$$\begin{aligned} \|(\mathcal{K}(y))(t) - (\mathcal{K}(y))(s)\|_V &\leq \|U_Y(t, 0)x - U_Y(s, 0)x\|_V \\ &+ \left\| \int_0^t U_Y(t, r)F(r, y(r))dr - \int_0^s U_Y(s, r)F(r, y(r))dr \right\|_V = I_1 + I_2. \end{aligned}$$

Note that applying (3.2) and by the continuity of trajectories of  $\{B_t^H, t \in [0, T]\}$

$$\begin{aligned} \sup_{t \in [0, T]} \|S_B(B_t^H(\omega))\|_{\mathcal{L}(V)} &\leq M_B \exp\{\omega_B \|B^H(\omega)\|_{\mathcal{C}([0, T])}\} \leq C_B(\omega), \\ \sup_{s, t \in [0, T]} \|S_B(B_t^H(\omega) - B_s^H(\omega))\|_{\mathcal{L}(V)} &\leq M_B \exp\{2\omega_B \|B^H(\omega)\|_{\mathcal{C}([0, T])}\} \leq C_B(\omega) \end{aligned} \quad (4.3)$$

hold for some constant  $0 < C_B(\omega) < +\infty$  depending on  $\omega \in \Omega$ .

By the strong continuity of  $S_B$  and  $U(\cdot, 0)$  on  $V$  it follows

$$\begin{aligned} I_1 &= \|U_Y(t, 0)x - U_Y(s, 0)x\|_V \\ &\leq \|(S_B(B_t^H) - S_B(B_s^H))U(t, 0)x\|_V + \|S_B(B_s^H)(U(t, 0) - U(s, 0))x\|_V \\ &\leq \|(S_B(B_t^H) - S_B(B_s^H))U(t, 0)x\|_V + C_B(\omega)\|U(t, 0) - U(s, 0)\|_V \xrightarrow{s \rightarrow t} 0. \end{aligned}$$

Now, let  $t > s$ . Then

$$\begin{aligned} I_2 &= \left\| \int_0^t U_Y(t, r)F(r, y(r))dr - \int_0^s U_Y(s, r)F(r, y(r))dr \right\|_V \\ &\leq \left\| \int_0^s (U_Y(t, r) - U_Y(s, r))F(r, y(r))dr \right\|_V + \left\| \int_s^t U_Y(t, r)F(r, y(r))dr \right\|_V = J_1 + J_2. \end{aligned}$$

Using (4.3), (3.4) and (4.5)

$$\begin{aligned} J_2 &= \left\| \int_s^t U_Y(t, r)F(r, y(r))dr \right\|_V \leq \int_s^t C_U \|S_B(B_t^H - B_r^H)\|_{\mathcal{L}(V)} \|F(r, y(r))\|_V dr \\ &\leq C_U C_B(\omega)(1 + \|y\|_{\mathcal{C}([0, T]; V)}) \int_s^t \bar{C}(r)dr \longrightarrow 0 \end{aligned}$$

as  $s \rightarrow t-$  or  $t \rightarrow s+$ .

Also

$$\begin{aligned} J_1 &= \left\| \int_0^s (U_Y(t, r) - U_Y(s, r))F(r, y(r))dr \right\|_V \\ &\leq \left\| \int_0^s (S_B(B_t^H - B_r^H) - S_B(B_s^H - B_r^H))U(t - r, 0)F(r, y(r))dr \right\|_V \\ &\quad + \left\| \int_0^s S_B(B_s^H - B_r^H)(U(t - r, 0) - U(s - r, 0))F(r, y(r))dr \right\|_V = K_1 + K_2. \end{aligned}$$

Since for any fixed  $0 \leq r \leq s$

$$\|(U(t - r, 0) - U(s - r, 0))F(r, y(r))\|_V \longrightarrow 0$$

as  $s \rightarrow t-$  or  $t \rightarrow s+$  and by (3.4)

$$\begin{aligned} \|(U(t-r, 0) - U(s-r, 0))F(r, y(r))\|_V &\leq 2C_U \|F(r, y(r))\|_V \\ &\leq 2C_U(1 + \|y\|_{\mathcal{C}([0, T]; V)})\bar{C}(r) \in L^1([0, T]), \end{aligned}$$

the convergence

$$\begin{aligned} K_2 &= \left\| \int_0^s S_B(B_s^H - B_r^H)(U(t-r, 0) - U(s-r, 0))F(r, y(r))dr \right\|_V \\ &\leq C_B(\omega) \int_0^s \|(U(t-r, 0) - U(s-r, 0))F(r, y(r))\|_V dr \rightarrow 0 \end{aligned}$$

is obtained as  $s \rightarrow t-$  or  $t \rightarrow s+$  by the Lebesgue dominated convergence theorem. Note that the set

$$K := \left\{ \bar{y} \in V; \exists 0 \leq s_1 \leq t_1 \leq T \quad \bar{y} = \int_0^{s_1} S_B(-B_r^H)U(t_1-r, 0)F(r, y(r))dr \right\}$$

is compact (being a continuous image of a compact set) and

$$\lim_{t \rightarrow s} \sup_{z \in K} \|(S_B(B_t^H) - S_B(B_s^H))z\|_V = 0.$$

Therefore

$$\begin{aligned} K_1 &= \left\| \int_0^s (S_B(B_t^H - B_r^H) - S_B(B_s^H - B_r^H))U(t-r, 0)F(r, y(r))dr \right\|_V \\ &= \left\| (S_B(B_t^H) - S_B(B_s^H)) \int_0^s S_B(-B_r^H)U(t-r, 0)F(r, y(r))dr \right\|_V \\ &\leq \sup_{z \in K} \|(S_B(B_t^H) - S_B(B_s^H))z\|_V \rightarrow 0 \end{aligned}$$

as  $s \rightarrow t-$  or  $t \rightarrow s+$ . Thus

$$\|(\mathcal{K}(y))(t) - (\mathcal{K}(y))(s)\|_V \rightarrow 0$$

as  $s \rightarrow t-$  or  $t \rightarrow s+$  and the function  $t \mapsto (\mathcal{K}(y))(t)$  is continuous on the interval  $[0, T]$  for any  $y \in \mathcal{C}([0, T]; V)$ .

For any  $y_1, y_2 \in \mathcal{C}([0, T]; V)$ ,  $t \in [0, T]$  and  $T > 0$  small enough there exists a constant  $0 < L_T(\omega) < 1$  such that

$$\begin{aligned} \|(\mathcal{K}(y_1))(t) - (\mathcal{K}(y_2))(t)\|_V &= \left\| \int_0^t U_Y(t, r)(F(r, y_1(r)) - F(r, y_2(r)))dr \right\|_V \\ &\leq C_B(\omega)C_U \int_0^t \|F(r, y_1(r)) - F(r, y_2(r))\|_V dr \\ &\leq C_B(\omega)C_U \|y_1 - y_2\|_{\mathcal{C}([0, T]; V)} \int_0^T \bar{L}(r)dr \leq L_T(\omega) \|y_1 - y_2\|_{\mathcal{C}([0, T]; V)} \end{aligned}$$

holds so that  $\mathcal{K}$  is a contraction mapping. Hence, by the Banach fixed-point theorem there exists a unique solution to the equation (4.2) for  $T$  small enough. Applying standard methods a unique continuous solution to (4.2) for any  $T > 0$  can be obtained.  $\square$

Consider an equation with a nonlinear perturbation of a drift part

$$\begin{aligned} dX_t &= AX_t dt + F(t, X_t)dt + BX_t dB_t^H, \\ X_0 &= x \in V. \end{aligned} \tag{4.4}$$



**Definition 4.2.** A  $(\mathcal{B}([0, T]) \otimes \mathcal{F})$ -measurable process  $\{X_t, t \in [0, T]\}$  is a **weak solution** to the equation (4.4) if for any  $y \in D^*$

$$\langle X_t, y \rangle_V = \langle x, y \rangle_V + \int_0^t \langle X_r, A^* y \rangle_V dr + \int_0^t \langle F(r, X_r), y \rangle_V dr + \int_0^t \langle X_r, B^* y \rangle_V dB_r^H \quad \mathbb{P}-a.s.$$

for all  $t \in [0, T]$ , where the integrals have to be well-defined.

*Remark 4.3.* (i) The conditions (i)<sub>F</sub> and (ii)<sub>F</sub> imply

$$\|F(t, x)\|_V \leq \bar{C}(t)(1 + \|x\|_V), \quad x \in V, \quad t \in [0, T]. \quad (4.5)$$

for a function  $\bar{C} \in L^1([0, T])$ .

(ii) In the Wiener case  $H = 1/2$  the solution to the equation (4.2) is the so-called mild solution to the equation

$$\begin{aligned} dX_t &= AX_t dt + F(t, X_t) dt + BX_t dW_t, \\ X_0 &= x \in V. \end{aligned}$$

In this case, S. Bonaccorsi ([3]) has shown that the solution to the equation (4.2) is also the weak one. This in general is not true for the equation (4.4) as is shown in the simple counterexample below.

**Example 4.4.** Consider a one-dimensional equation

$$dX_t = aX_t dt + bX_t dB_t^H, \quad X_0 = 1, \quad (4.6)$$

where  $a, b \in \mathbb{R}$  are nonzero constants. Note that the equation (4.6) takes the form (4.4) with  $F(t, x) = ax$ ,  $A = 0$ ,  $B = bI$  and  $x_0 = 1$ .

The solution to the equation (4.6) is given by the formula

$$X_t = \exp \left\{ bB_t^H - \frac{1}{2}b^2t^{2H} + at \right\}, \quad t \in [0, T],$$

and the random evolution system corresponding to the above choice of coefficients is

$$U_Y(t, s) = S_B(B_t^H - B_s^H)U(t, s) = \exp \left\{ b(B_t^H - B_s^H) - \frac{1}{2}b^2(t - s)^{2H} \right\}, \quad 0 \leq s \leq t \leq T.$$

It is now easy to compute that the solution  $\{X_t, t \in [0, T]\}$  DOES NOT satisfy the mild formula

$$y(t) = U_Y(t, 0) + \int_0^t U_Y(t, r)F(r, y(r))dr. \quad (4.7)$$

Note that if we define the system  $\{\bar{U}_Y(t, s), 0 \leq s \leq t \leq T\}$  as

$$\bar{U}_Y(t, s) = S_B(B_t^H - B_s^H)U(t, s) = \exp \left\{ b(B_t^H - B_s^H) - \frac{1}{2}b^2(t^{2H} - s^{2H}) \right\}, \quad 0 \leq s \leq t \leq T,$$

the above mild formula holds if  $U_Y$  is replaced by  $\bar{U}_Y$ .

*Remark 4.5.* Let the assumptions of Theorem 3.4 be satisfied. Then the system  $\{\bar{U}_Y(t, s), 0 \leq s \leq t \leq T\}$  defined as

$$\bar{U}_Y(t, s)x = S_B(B_t^H - B_s^H)U(t, s)x, \quad x \in V, \quad 0 \leq s \leq t \leq T, \quad (4.8)$$

is a weak solution to the equation

$$\begin{aligned} dY_t &= A(t)Y_t dt + H((t-s)^{2H-1} - t^{2H-1})B^2Y_t dt + BY_t dB_t^H, \quad t > s, \\ Y_s &= x. \end{aligned}$$

This result can be obtained in the same way as Theorem 3.4. The system  $\{\bar{U}_Y(t, s), 0 \leq s \leq t \leq T\}$  defined in Example 4.4 is a particular case of (4.8). Moreover, this system has a composition property.

It is easy to verify the fact that  $\{U_Y(t, s), 0 \leq s \leq t \leq T\}$  does not possess the composition property, which means that the equation (3.1) does not define a cocycle in the usual way. On the other hand, in [2] it has been proved (for the case of stochastic equation with homogeneous right hand side and bilinear fractional noise) that the cocycle property does hold in the case when stochastic integration in Stratonovich sense is considered.

The natural question is whether there is a chance to obtain a weak solution as the unique solution to the equation (4.2). The positive answer gives the next theorem but only under the restrictive assumption on  $F$  that it does not depend on the space variable.

**Theorem 4.6.** *Assume that the measurable function  $F : [0, T] \rightarrow V$  is affine and that  $\|F\|_V \in L^2([0, T])$ . Then the unique continuous solution  $\{X_t, t \in [0, T]\}$  to the equation*

$$X_t^M = U_Y(t, 0)x + \int_0^t U_Y(t, r)F(r)dr \quad (4.9)$$

*stated in Theorem 4.1 is a weak solution to the equation*

$$\begin{aligned} dX_t &= AX_t dt + F(t)dt + BX_t dB_t^H, \\ X_0 &= x \in V. \end{aligned} \quad (4.10)$$

The main idea of the proof is to use standard and stochastic Fubini theorem for the Skorokhod integral stated in [11], Lemma 2.10, or [13], Exercise 3.2.8.

**Lemma 4.7.** *Consider a random field  $\{u(t, x), t \in [0, T], x \in G\}$ , where  $G \subset \mathbb{R}$  is a bounded set, such that*

- (i)<sub>W</sub>  $u \in L^2(\Omega \times [0, T] \times G)$ ,
- (ii)<sub>W</sub>  $u(\cdot, x) \in \text{Dom } \delta_W$  for a.e.  $x \in G$ ,
- (iii)<sub>W</sub>  $\mathbb{E} \left[ \int_G \left( \int_0^T u(t, x) dW_t \right)^2 dx \right] < +\infty$ .

*Then the process  $\{\int_G u(t, x)dx, t \in [0, T]\} \in \text{Dom } \delta_W$  and*

$$\int_0^T \left( \int_G u(t, x)dx \right) dW_t = \int_G \left( \int_0^T u(t, x) dW_t \right) dx.$$

Due to the relationship between Skorokhod integral with respect to Wiener process and fractional Brownian motion (see (2.6) or [14] for more details) (ii)<sub>W</sub>, (iii)<sub>W</sub> are equivalent to

- (ii)<sub>H</sub>  $u_H(\cdot, x) \in \text{Dom } \delta_H$  for a.e.  $x \in G$ ,
- (iii)<sub>H</sub>  $\mathbb{E} \left[ \int_G \left( \int_0^T u_H(t, x) dB_t^H \right)^2 dx \right] < +\infty$ ,

respectively, where  $u_H(t, x) = (\mathcal{K}_H^*)^{-1}(u(\cdot, x))(t), t \in [0, T]$ . The conclusion of Lemma 4.7 can be reformulated in the following way. The process  $\{\int_G u_H(t, x)dx, t \in [0, T]\} \in \text{Dom } \delta_H$  and

$$\int_0^T \left( \int_G u_H(t, x)dx \right) dB_t^H = \int_G \left( \int_0^T u_H(t, x) dB_t^H \right) dx.$$

The proof of Theorem (4.6) is based on the following lemma.

**Lemma 4.8.** *The equalities*

$$\int_0^t \int_0^r \langle U_Y(r, v)F(v), A^*\zeta \rangle_V dv dr = \int_0^t \int_v^t \langle U_Y(r, v)F(v), A^*\zeta \rangle_V dr dv \quad (4.11)$$

and

$$\int_0^t \int_0^r \langle U_Y(r, v)F(v), B^*\zeta \rangle_V dv dB_r^H = \int_0^t \int_v^t \langle U_Y(r, v)F(v), B^*\zeta \rangle_V dB_r^H dv$$

hold  $\mathbb{P}$ -a.s. for any  $t \in [0, T]$  and fixed  $\zeta \in D^*$ .

*Proof.* It is necessary to verify the assumptions of standard and stochastic Fubini theorem. Notice that the Fernique theorem (see e.g. [8]) yields that there exists a random variable  $C_{BH}(\omega)$  such that  $C_{BH} \in L^q(\Omega)$  for any  $q \in [1, +\infty)$  and

$$M_B \exp\{l\omega_B\|B^H(\omega)\|_{\mathcal{C}([0, T])}\} \leq C_{BH}(\omega), \quad \omega \in \Omega, l = 1, 2. \quad (4.12)$$

Since by (4.12) and (3.4)

$$\begin{aligned} \int_0^t \int_0^r |\langle U_Y(r, v)F(v), A^*\zeta \rangle_V| dv dr &\leq \int_0^T \int_0^T C_{BH}(\omega) C_U \|F(v)\|_V \|A^*\zeta\|_V dv dr \\ &\leq K(\omega) \int_0^T \|F(v)\|_V dv < +\infty \end{aligned}$$

for a.e.  $\omega \in \Omega$ , (4.11) follows by the standard Fubini theorem.

Denote

$$\begin{aligned} u_H(r, s) &= \langle U_Y(r, s)F(s), B^*\zeta \rangle_V, \quad 0 \leq s \leq r \leq t, \\ u(r, s) &= (\mathcal{K}_H^* u_H(\cdot, s))(r), \quad 0 \leq s \leq r \leq t, \end{aligned}$$

and verify that (i)<sub>W</sub>, (ii)<sub>H</sub> and (iii)<sub>H</sub> hold for the corresponding processes. First show that  $u \in L^2([0, t]^2 \times \Omega)$ . Using (2.2)

$$\begin{aligned} \mathbb{E} \left[ \int_0^t \int_0^t u^2(r, s) dr ds \right] &\leq K_e \mathbb{E} \left[ \int_0^t \int_0^t u_H^2(r, s) dr ds \right] \\ &\leq K_e \mathbb{E} \left[ \int_0^t \int_0^t (C_{BH}(\omega) C_U \|F(s)\|_V \|B^*\zeta\|_V)^2 dr ds \right] < +\infty, \end{aligned}$$

and (i)<sub>W</sub> follows. To prove (ii)<sub>H</sub> it suffices to show (in the view of (2.3)) that  $u_H(\cdot, s) \in \mathbb{D}_H^{1,2}(|\mathcal{H}|)$  for a.e.  $s \in [0, t]$  which is true whenever

$$\max \left\{ \sup_{r \in [0, t]} \mathbb{E} [u_H^2(r, s)], \sup_{r \in [0, t]} \sup_{v \in [0, t]} \mathbb{E} [(D_v^H u_H(r, s))^2] \right\} < +\infty \quad (4.13)$$

for a.e.  $s \in [0, t]$ . Since

$$D_v^H u_H(r, s) = \langle U_Y(r, s)F(s), (B^*)^2\zeta \rangle_V I_{(s, r]}(v)$$

the inequalities

$$\begin{aligned} &\sup_{r \in [0, t]} \sup_{v \in [0, t]} \mathbb{E} [(D_v^H u_H(r, s))^2] \\ &\leq \sup_{r \in [0, t]} \mathbb{E} [(C_{BH}(\omega) C_U \|F(s)\|_V \|(B^*)^2\zeta\|_V)^2] = K \|F(s)\|_V^2 < +\infty \end{aligned}$$

and

$$\sup_{r \in [0, t]} \mathbb{E} [u_H^2(r, s)] \leq \mathbb{E} [(C_{BH}(\omega) C_U \|F(s)\|_V \|B^* \zeta\|_V)^2] \leq K \|F(s)\|_V^2 < +\infty$$

hold for a.e.  $s \in [0, t]$  which completes the proof of (4.13).

Finally, applying the estimate on the Skorokhod integral (2.4) and the previous part of the proof of (4.13)

$$\begin{aligned} \mathbb{E} \left[ \int_0^t \left( \int_0^t u_H(r, s) dB_r^H \right)^2 ds \right] &= \int_0^t \mathbb{E} \left[ \left( \int_0^t u_H(r, s) dB_r^H \right)^2 \right] ds \\ &\leq C_{H,2} \int_0^t \left( \mathbb{E} [\|u_H(\cdot, s)\|_{L^2([0, t])}^2] + \mathbb{E} [\|D^H u_H(\cdot, s)\|_{L^2([0, t]^2)}^2] \right) ds \\ &\leq C_{H,2} \int_0^t (t + t^2) K \|F(s)\|_V^2 ds < +\infty \end{aligned}$$

holds and (iii)<sub>H</sub> follows.  $\square$

*Proof of Theorem 4.6.* Fix  $\zeta \in D^*$ . Since  $\{X_t, t \in [0, T]\}$  satisfies (4.9) and  $\{U_Y(t, s)x, s \leq t \leq T\}$  is a weak solution to the equation (4.1)

$$\begin{aligned} \int_0^t \langle X_r, A^* \zeta \rangle_V dr + \int_0^t \langle X_r, B^* \zeta \rangle_V dB_r^H &= \int_0^t \langle U_Y(r, 0)x, A^* \zeta \rangle_V dr \\ &+ \int_0^t \int_0^r \langle U_Y(r, v)F(v), A^* \zeta \rangle_V dv dr + \int_0^t \langle U_Y(r, 0)x, B^* \zeta \rangle_V dB_r^H \\ &+ \int_0^t \int_0^r \langle U_Y(r, v)F(v), B^* \zeta \rangle_V dv dB_r^H \\ &= \langle U_Y(t, 0)x, \zeta \rangle_V - \langle x, \zeta \rangle_V + \int_0^t \int_v^t \langle U_Y(r, v)F(v), A^* \zeta \rangle_V dr dv \\ &+ \int_0^t \int_v^t \langle U_Y(r, v)F(v), B^* \zeta \rangle_V dB_r^H dv \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

holds for any  $t \in [0, T]$ , where in the last equality Lemma 4.8 is used.

Applying again that  $\{U_Y(t, s)x, s \leq t \leq T\}$  is a weak solution to the equation (4.1)

$$\begin{aligned} \int_0^t \langle X_r, A^* \zeta \rangle_V dr + \int_0^t \langle X_r, B^* \zeta \rangle_V dB_r^H \\ &= \langle U_Y(t, 0)x, \zeta \rangle_V - \langle x, \zeta \rangle_V + \int_0^t \langle U_Y(t, v)F(v), \zeta \rangle_V dv - \int_0^t \langle F(v), \zeta \rangle_V dv \\ &= \langle X_t, \zeta \rangle_V - \langle x, \zeta \rangle_V - \int_0^t \langle F(v), \zeta \rangle_V dv \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

is obtained for any  $t \in [0, T]$  and the conclusion follows.  $\square$

*Remark 4.9.* In view of Example 4.4, one can ask whether the solution to the equation (4.2) is a weak one to some equation. A partial answer is given by the subsequent Theorem the proof of which is similar to the one of Theorem 4.6 (but more technical) and is omitted.

*Theorem 4.10.* Let the assumptions of Theorem 4.1 hold and  $\{X_t, t \in [0, T]\}$  be the solution to the equation (4.2) such that there exists a constant  $C_X < +\infty$

$$\max \left\{ \sup_{t \in [0, T]} \mathbb{E} \|X_t\|_V^4, \sup_{t \in [0, T]} \sup_{v \in [0, T]} \mathbb{E} \|D_v^H X_t\|_V^4 \right\} \leq C_X. \quad (4.14)$$

In addition, let  $F$  be Fréchet differentiable with respect to the space variable for any time  $t \in [0, T]$ . Suppose that there exists a function  $C \in L^4([0, T])$  such that

$$\max\{\|F(t, x)\|_V, \|F'_x(t, x)\|\} \leq C(t), \quad t \in [0, T], \quad (4.15)$$

holds. Then  $\{X_t, t \in [0, T]\}$  is a solution to the integral equation

$$\begin{aligned} X_t = x &+ \int_0^t A X_r dr + \int_0^t F(r, X_r) dr + \int_0^t B X_r dB_r^H \\ &+ \int_0^t \alpha_H \int_0^T \int_r^t |v - w|^{2H-2} B U_Y(v, r) F'_x(r, X_r) D_w^H X_r dv dw dr \end{aligned}$$

in a weak sense, i.e. for any  $y \in D^*$

$$\begin{aligned} \langle X_t, y \rangle_V &= \langle x, y \rangle_V + \int_0^t \langle X_r, A^* y \rangle_V dr + \int_0^t \langle F(r, X_r), y \rangle_V dr + \int_0^t \langle X_r, B^* y \rangle_V dB_r^H \\ &+ \int_0^t \alpha_H \int_0^T \int_r^t |v - w|^{2H-2} \langle U_Y(v, r) F'_x(r, X_r) D_w^H X_r, B^* y \rangle_V dv dw dr \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

for all  $t \in [0, T]$ .

*Remark 4.11.* The condition (4.14) implies that  $X \in \mathbb{D}_H^{1,4}(|\mathcal{H}|)$ .

**Example 4.12.** Consider the stochastic parabolic equation of the second order with the additional affine term in a drift part

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= (Lu(t, \cdot))(x) + f(t, x) + bu(t, x) \frac{dB^H}{dt}, \\ u(0, x) &= x_0(x), \quad x \in \mathcal{O}, \\ u(t, x) &= 0, \quad (t, x) \in [0, T] \times \partial\mathcal{O}, \end{aligned} \quad (4.16)$$

where  $\mathcal{O} \subset \mathbb{R}^d$  is a bounded domain with the boundary of class  $\mathcal{C}^2$ ,  $b \in \mathbb{R} \setminus \{0\}$  and

$$(Lu(t, \cdot))(x) = a_0(x)u(t, x) + \sum_{i=1}^d a_i(x) \frac{\partial u}{\partial x_i}(t, x) + \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(t, x)$$

is a strongly elliptic operator on  $\mathcal{O}$ .

Suppose that the functions  $a_0, a_i, a_{ij} \in \mathcal{C}^\infty(\bar{\mathcal{O}})$  for  $i, j = 1, \dots, d$ . Let  $V = L^2(\mathcal{O})$ . Assume that the mapping  $F : [0, T] \rightarrow V$ ;  $F(t) := f(t, \cdot)$ , satisfies  $F \in L^2([0, T]; V)$ .

Equation (4.16) can be rewritten in the form (4.10), where

$$(Au(t, \cdot))(x) = (Lu(t, \cdot))(x),$$

with  $\text{Dom}(A) = D = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$  and  $B = bI \in \mathcal{L}(V)$ . The adjoint operator  $A^*$  has the same form as the operator  $A$  (possibly, with different coefficients), hence  $\text{Dom}(A^*) = D$ .

In this case the assumptions of Theorem 4.6 (including those of Theorems 3.4 and 4.1) are satisfied, therefore the process  $\{X_t, t \in [0, T]\}$  defined as

$$X_t = U_Y(t, 0)x_0 + \int_0^t U_Y(t, r)F(r)dr$$

is a weak solution to the equation (4.16). Note that the process  $\{U_Y(t, s), 0 \leq s \leq t \leq T\}$  defined in Theorem 3.4 has the form

$$U_Y(t, s) = \exp \left\{ b(B_t^H - B_s^H) - \frac{1}{2}b^2(t-s)^{2H} \right\} S_L(t-s), \quad 0 \leq s \leq t \leq T,$$

where  $\{S_L(t), t \in [0, T]\}$  is the strongly continuous semigroup on  $V$  generated by operator  $A$ .

Theorem 4.6 may serve as a useful tool for analysis of a behaviour of the weak solutions to (4.10). As an example a simple result on large-time behaviour of the solution to the equation

$$\begin{aligned} dX_t &= (AX_t + F(t))dt + bX_t dB_t^H, \quad t > 0, \\ X_0 &= x, \end{aligned} \tag{4.17}$$

is provided, where  $A : \text{Dom}(A) \subset V \rightarrow V$  is the generator of a strongly continuous semigroup  $\{S_A(t), t \geq 0\}$  and  $b \in \mathbb{R} \setminus \{0\}$ .

It is easily seen that

$$U_Y(t, s) = \exp \left\{ b(B_t^H - B_s^H) - \frac{1}{2}b^2(t-s)^{2H} \right\} S_A(t-s), \quad 0 \leq s \leq t < +\infty,$$

and since there exist some constants  $M > 0, \omega \in \mathbb{R}$ , such that

$$\|S_A(t)\|_{\mathcal{L}(V)} \leq Me^{\omega t}, \quad t \geq 0,$$

the inequality

$$\|U_Y(t, s)\|_{\mathcal{L}(V)} \leq M \exp \left\{ b(B_t^H - B_s^H) - \frac{1}{2}b^2(t-s)^{2H} + \omega(t-s) \right\}, \quad 0 \leq s \leq t < +\infty, \tag{4.18}$$

is obtained.

**Proposition 4.13.** *Assume that  $F \in L^2([0, T]; V)$ . Then the solution  $\{X_t, t \geq 0\}$  to the equation (4.17) satisfies*

$$\|X_t\|_V \leq y(t), \quad t \geq 0, \quad \mathbb{P}\text{-a.s.},$$

where  $y$  is a solution to one-dimensional equation

$$\begin{aligned} dy(t) &= (\omega y(t) + \|F(t)\|_V)dt + by(t)dB_t^H, \quad t > 0, \\ y(0) &= M\|x\|_V. \end{aligned} \tag{4.19}$$

*Proof.* The proof easily follows from (4.9) and (4.18) because

$$\begin{aligned} \|X_t\|_V &\leq M \exp \left\{ bB_t^H - \frac{1}{2}b^2t^{2H} + \omega t \right\} \|x\|_V \\ &\quad + \int_0^t \exp \left\{ b(B_t^H - B_s^H) - \frac{1}{2}b^2(t-s)^{2H} + \omega(t-s) \right\} M\|F(s)\|_V ds \end{aligned} \tag{4.20}$$

and by Theorem 4.6 the right-hand side of (4.20) is exactly the formula for the solution to (4.19).  $\square$

**Corollary 4.14.** *For each  $p \geq 1$  there exists a constant  $c_p > 0$  depending only on  $p$  such that*

$$\begin{aligned} \mathbb{E} [\|X_t\|_V^p] &\leq c_p M \exp \left\{ \frac{(p^2 - p)b^2}{2} t^{2H} + p\omega t \right\} \|x\|_V^p \\ &\quad + Mt^{p-1} \int_0^t \exp \left\{ \frac{(p^2 - p)b^2}{2} (t-s)^{2H} + p\omega(t-s) \right\} \|F(s)\|_V^p ds, \quad t \geq 0. \end{aligned} \quad (4.21)$$

*In particular, if  $F(t) \equiv F$  does not depend on  $t \geq 0$ , for each  $\epsilon > 0$  there exists  $C_\epsilon > 0$  such that*

$$\mathbb{E} [\|X_t\|_V^p] \leq C_\epsilon \exp\{(\hat{c} + \epsilon)t^{2H}\}, \quad t \geq 0, \quad (4.22)$$

*holds with  $\hat{c} = 1/2b^2(p^2 - p)$ .*

*Proof.* The inequality (4.21) easily follows from (4.20) if we take into account that

$$\mathbb{E} \left[ \exp \left\{ p \left( b(B_t^H - B_s^H) - \frac{1}{2}b^2(t-s)^{2H} + \omega(t-s) \right) \right\} \right] = \exp \{ \hat{c}(t-s)^{2H} + p\omega(t-s) \}$$

for all  $0 \leq s \leq t$  and apply the Hölder inequality on the second term on the right-hand side of (4.20). The inequality (4.22) is an immediate consequence of (4.21).  $\square$

*Remark 4.15.* A simple one-dimensional example shows that the bound  $\hat{c}$  in (4.22) is, in some sense, sharp. Take  $V = \mathbb{R}$ ,  $A = \omega$ ,  $F = 0$ ,  $x \neq 0$ , then

$$|X_t|^p = |x|^p \exp \left\{ p\omega t - \frac{1}{2}b^2 p t^{2H} + p b B_t^H \right\}, \quad t \geq 0, \quad p > 1,$$

hence for each  $\epsilon > 0$  there exists  $\tilde{C}_\epsilon > 0$  such that

$$\mathbb{E} [|X_t|^p] = |x|^p \exp \{ \hat{c} t^{2H} + p\omega t \} \geq \tilde{C}_\epsilon \exp\{(\hat{c} - \epsilon)t^{2H}\}, \quad t \geq 0.$$

It means that for  $p > 1$  the  $p$ -th moment of the solution to linear equation may be destabilized by adding bilinear fractional noise of the form  $bX_t \dot{B}_t^H$ ,  $b \neq 0$ , even if the original equation is stable (here  $\omega < 0$ ). It may be interesting to note that from [4], Remark 3.7, applied to the same example it follows that the solution tends to zero pathwise exponentially fast as  $t \rightarrow +\infty$ , even if the equation without noise is not stable (i.e.  $\omega > 0$ ).

## 5. UNIQUENESS OF MILD SOLUTION

This section is devoted to the proof of the uniqueness of the mild solution to the equation

$$dX_t = AX_t dt + BX_t dB_t^H, \quad X_0 = x, \quad (5.1)$$

on the interval  $[0, T]$ . Let  $H > 1/2$  and recall that  $S_B$  is a strongly continuous group generated by  $B$  and  $U$  is a strongly continuous evolution system associated with operators  $A - Ht^{2H-1}B^2$ ,  $t \in [0, T]$ .

**Theorem 5.1.** *Let the conditions (A1), (A2), (AB) be satisfied and let  $B \in \mathcal{L}(V)$ . Then the process  $X = \{X_t, t \in [0, T]\}$  given by*

$$X_t = S_B(B_t^H)U(t, 0)x \quad (5.2)$$

*is a mild solution to the equation (5.1), i.e.*

$$X_t = S_A(t)x + \int_0^t S_A(t-r)BX_r dB_r^H \quad \mathbb{P}\text{-a.s.}$$

*for all  $t \in [0, T]$ , where  $\{S_A(t), t \geq 0\}$  is an analytic semigroup generated by  $A$ .*

*Proof.* See [7]. □

The aim is to show, that  $X$  defined by (5.2) is a unique mild solution to (5.1). The idea of the proof is to use the fractional Wiener chaos decomposition as in the paper [16], where the result is proved in a one-dimensional case.

The construction of multiple fractional integrals and fractional Wiener chaos decomposition that are used below, are made only for real-valued random variables. However, all remains true for Hilbert space-valued random variables (see e.g. [12]). For the simplicity the Hilbert space-valued notation is the same as the real-valued notation.

Let  $\mathcal{H}^{\otimes n}$  denote the  $n$ th tensor product of  $\mathcal{H}$  for any  $n \geq 2$ . Set  $\mathcal{H}^{\otimes 1} \equiv \mathcal{H}$  and  $\mathcal{H}^{\otimes 0} \equiv \mathbb{R}$  or  $V$ , respectively.

**Definition 5.2.** Let  $n \in \mathbb{N}$ . For  $f \in \mathcal{H}^{\otimes n}$  symmetric the multiple fractional integral of order  $n$  of  $f$  is defined as

$$I_n^H(f) = \delta_H^n(f),$$

where  $\delta_H^n$  is the multiple divergence operator (Skorokhod integral) of order  $n$  (for the definition see e.g. [12]).

Note that  $\delta_H^1 \equiv \delta_H$ .

As in the Wiener case the functions  $F \in L^2(\Omega; \mathcal{G}, \mathbb{P})$  (where  $\mathcal{G}$  denotes the  $\sigma$ -field generated by  $\{B_t^H, t \in [0, T]\}$ ) admit the unique fractional Wiener chaos decomposition

$$F = \mathbb{E}[F] + \sum_{n=1}^{+\infty} I_n^H(f_n),$$

where  $f_n \in \mathcal{H}^{\otimes n}$  are symmetric elements which are uniquely determined (see [12] or [13]). Let

$$\mathcal{H}_n = I_n^H(\mathcal{H}^{\otimes n})$$

be the fractional Wiener chaos of order  $n$ .

**Theorem 5.3.** Under the assumptions of Theorem 5.1 the mild solution

$$\{X_t = S_B(B_t^H)U(t, 0)x, \ t \in [0, T]\},$$

to the equation (5.1) is unique in  $\text{Dom } \delta_H$ .

*Proof.* Clearly,  $X = \{X_t, t \in [0, T]\} \in \mathbb{D}_H^{1,2}(|\mathcal{H}|) \subset \text{Dom } \delta_H$ . Take another mild solution  $Y = \{Y_t, t \in [0, T]\} \in \text{Dom } \delta_H$  to the equation (5.1). Then the processes  $X$  and  $Y$  satisfy

$$\begin{aligned} X_t &= S_A(t)x + \int_0^t S_A(t-r)BX_r dB_r^H, \\ Y_t &= S_A(t)x + \int_0^t S_A(t-r)BY_r dB_r^H, \end{aligned}$$

respectively. Define the process  $Z = \{Z_t, t \in [0, T]\}$  as

$$Z_t = X_t - Y_t, \ t \in [0, T].$$

Let

$$Z_t = \sum_{n=0}^{+\infty} I_n(z_n(t, \cdot))$$



be the fractional Wiener chaos decomposition of process  $Z$ , where  $z_n(t, \cdot) \in \mathcal{H}^{n+1}$  be the symmetric elements in the last  $n$  variables. Since

$$z_0(t) = I_0(z_0(t)) = \mathbb{E}[Z_t] = \mathbb{E}[X_t - Y_t] = S_A(t)x - S_A(t)x = 0$$

for all  $t \in [0, T]$ , we get

$$Z_t = \sum_{n=1}^{+\infty} I_n(z_n(t, \cdot)).$$

The definition of Skorokhod integral via multiple integrals yields

$$\begin{aligned} \sum_{n=1}^{+\infty} I_n(z_n(t, \cdot)) &= Z_t = \int_0^t S_A(t-r) B Z_r dB_r^H = \int_0^t \sum_{n=0}^{+\infty} I_n(S_A(t-r) B z_n(r, \cdot)) dB_r^H \\ &= \sum_{n=0}^{+\infty} I_{n+1}(\text{Sym}(S_A(t-\cdot) B z_n(\cdot))) = \sum_{n=1}^{+\infty} I_n(\text{Sym}(S_A(t-\cdot) B z_{n-1}(\cdot))), \end{aligned}$$

where  $\text{Sym}(f)$  denotes the symmetrization of  $f$  in all variables. From the uniqueness of Wiener chaos expansion it follows

$$z_n(t, \cdot) = \text{Sym}(S_A(t-\cdot) B z_{n-1}(\cdot)), \quad n \geq 1.$$

Since  $z_0 \equiv 0$  we obtain by induction that

$$z_1 \equiv 0, z_2 \equiv 0, \dots$$

hence  $Z \equiv 0$  and the proof is completed.  $\square$

## 6. UNIQUENESS OF WEAK SOLUTION

Let  $\mathcal{M}$  be the space of  $\mathcal{B}([0, T]) \otimes \mathcal{F}$ -measurable processes  $Z : [0, T] \times \Omega \rightarrow V$  with continuous trajectories such that

$$Z \in \text{Dom } \delta_H \quad \text{and} \quad \mathbb{E} \sup_{t \in [0, T]} \|Z_t\|_V^2 < +\infty.$$

The aim is to show that a weak solution to the equation

$$dX_t = (AX_t + F(t))dt + BX_t dB_t^H, \quad X_0 = x, \quad (6.1)$$

is unique in the space  $\mathcal{M}$ . On this purpose the following version of integration by parts formula is necessary.

**Lemma 6.1.** *Let  $Y \in \mathcal{M}$  be a weak solution to the equation*

$$dY_t = AY_t dt + BY_t dB_t^H, \quad Y_0 = 0.$$

*Then*

$$\langle Y_t, \zeta(t) \rangle_V = \int_0^t \langle Y_r, A^* \zeta(r) + \zeta'(r) \rangle_V dr + \int_0^t \langle Y_r, B^* \zeta(r) \rangle_V dB_r^H \quad (6.2)$$

*for any  $\zeta \in \mathcal{C}^1([0, T]; D^*)$ .*

*Proof. 1st step:* Let  $\zeta$  have the form

$$\zeta(t) = \varphi(t)\xi, \quad \varphi \in \mathcal{C}^1([0, T]), \xi \in D^*. \quad (6.3)$$

Let  $\{t_k, k = 0, \dots, n\}$  be the partition of interval  $[0, t]$ . Then

$$\begin{aligned} \langle Y_t, \zeta(t) \rangle_V &= \varphi(t) \langle Y_t, \xi \rangle_V = \sum_{k=0}^{n-1} (\varphi(t_{k+1}) \langle Y_{t_{k+1}}, \xi \rangle_V - \varphi(t_k) \langle Y_{t_k}, \xi \rangle_V) \\ &= \sum_{k=0}^{n-1} (\varphi(t_{k+1}) - \varphi(t_k)) \langle Y_{t_{k+1}}, \xi \rangle_V + \sum_{k=0}^{n-1} \varphi(t_k) (\langle Y_{t_{k+1}}, \xi \rangle_V - \langle Y_{t_k}, \xi \rangle_V) = S_1 + S_2. \end{aligned} \quad (6.4)$$

Since

$$\left| \sum_{k=0}^{n-1} (\varphi(t_{k+1}) - \varphi(t_k)) \langle Y_{t_{k+1}}, \xi \rangle_V \right| \leq \|\varphi\|_{C^1([0, T])} \sup_{r \in [0, T]} \|Y_r\|_V \left( \sum_{k=0}^{n-1} (t_{k+1} - t_k) \right) \|\xi\|_V$$

and

$$\mathbb{E} \left[ \sup_{r \in [0, T]} \|Y_r\|_V^2 \right] < +\infty,$$

it follows

$$S_1 = \sum_{k=0}^{n-1} (\varphi(t_{k+1}) - \varphi(t_k)) \langle Y_{t_{k+1}}, \xi \rangle_V \xrightarrow{n \rightarrow +\infty} \int_0^t \varphi'(r) \langle Y_r, \xi \rangle_V dr \quad \text{in } L^2(\Omega)$$

in virtue of the Lebesgue dominated convergence theorem. The second sum  $S_2$  can be split into two summands

$$S_2 = \sum_{k=0}^{n-1} \varphi(t_k) \left( \int_{t_k}^{t_{k+1}} \langle Y_r, A^* \xi \rangle_V dr + \int_{t_k}^{t_{k+1}} \langle Y_r, B^* \xi \rangle_V dB_r^H \right) = S_{21} + S_{22}.$$

The first summand

$$S_{21} = \sum_{k=0}^{n-1} \varphi(t_k) \int_{t_k}^{t_{k+1}} \langle Y_r, A^* \xi \rangle_V dr \xrightarrow{n \rightarrow +\infty} \int_0^t \varphi(r) \langle Y_r, A^* \xi \rangle_V dr \quad \text{in } L^2(\Omega)$$

by the Lebesgue dominated convergence theorem because

$$\left| \sum_{k=0}^{n-1} \varphi(t_k) \int_{t_k}^{t_{k+1}} \langle Y_r, A^* \xi \rangle_V dr \right| \leq \|\varphi\|_{C^1([0, T])} \sup_{r \in [0, T]} \|Y_r\|_V \|A^* \xi\|_V T.$$

Since  $Y_t \in L^2(\Omega)$  satisfies (6.4) we conclude that

$$S_{22} = \sum_{k=0}^{n-1} \varphi(t_k) \int_{t_k}^{t_{k+1}} \langle Y_r, B^* \xi \rangle_V dB_r^H = \int_0^t \sum_{k=0}^{n-1} \varphi(t_k) I_{(t_k, t_{k+1}]}(r) \langle Y_r, B^* \xi \rangle_V dB_r^H$$

converges to a random variable denoted by  $Y_t^1$  in  $L^2(\Omega)$  as  $n \rightarrow +\infty$ . It remains to show that  $Y_t^1 = \int_0^t \varphi(r) \langle Y_r, B^* \xi \rangle_V dB_r^H$  by the closedness of Skorokhod integral. Denote

$$\Phi_n(r) = \sum_{k=0}^{n-1} \varphi(t_k) I_{(t_k, t_{k+1}]}(r) \langle Y_r, B^* \xi \rangle_V, \quad r \in [0, t], n \in \mathbb{N}.$$

Then  $\Phi_n \in \text{Dom } \delta_H$ ,

$$\Phi_n(r) \xrightarrow{n \rightarrow +\infty} \varphi(r) \langle Y_r, B^* \xi \rangle_V$$

for any fixed  $r, \omega$ , and

$$\begin{aligned} |\Phi_n(r)| &= \left| \sum_{k=0}^{n-1} \varphi(t_k) I_{(t_k, t_{k+1}]}(r) \langle Y_r, B^* \xi \rangle_V \right| \leq \|\varphi\|_{C^1([0, T])} \sup_{r \in [0, T]} \|Y_r\|_V \|B^* \xi\|_V \sum_{k=0}^{n-1} I_{(t_k, t_{k+1}]}(r) \\ &= \|\varphi\|_{C^1([0, T])} \sup_{r \in [0, T]} \|Y_r\|_V \|B^* \xi\|_V. \end{aligned}$$

By the Lebesgue dominated convergence theorem  $\Phi_n \in L^2(\Omega; L^2([0, t]; V))$  and

$$\Phi_n \xrightarrow{n \rightarrow +\infty} \varphi \langle Y, B^* \xi \rangle_V \quad \text{in } L^2(\Omega; L^2([0, t]; V)).$$

By the closedness of Skorokhod integral  $Y_t^1 = \int_0^t \varphi(r) \langle Y_r, B^* \xi \rangle_V dB_r^H$  and equality (6.2) holds for  $\zeta$  of the form (6.3).

**2nd step:** Let  $\zeta \in \mathcal{C}^1([0, T]; D^*)$ . Then there exists a sequence  $\{\zeta_n, n \in \mathbb{N}\} \subset \mathcal{C}^1([0, T]; D^*)$  of elementary functions of the form (6.3) such that  $\zeta_n \xrightarrow{n \rightarrow +\infty} \zeta$  in  $\mathcal{C}^1([0, T]; D^*)$ . The aim is to pass to the limit in the equation

$$\langle Y_t, \zeta_n(t) \rangle_V = \int_0^t \langle Y_r, A^* \zeta_n(r) + \zeta'(r) \rangle_V dr + \int_0^t \langle Y_r, B^* \zeta_n(r) \rangle_V dB_r^H$$

in  $L^2(\Omega)$ . Clearly,

$$|\langle Y_t, \zeta_n(t) - \zeta(t) \rangle_V| \leq \sup_{r \in [0, T]} \|Y_r\|_V \|\zeta_n - \zeta\|_{\mathcal{C}^1([0, T]; D^*)} \xrightarrow{n \rightarrow +\infty} 0$$

and

$$\begin{aligned} &\mathbb{E} \left[ \left( \int_0^t \langle Y_r, A^* (\zeta_n(r) - \zeta(r)) + (\zeta'_n(r) - \zeta'(r)) \rangle_V dr \right)^2 \right] \\ &\leq \mathbb{E} \left[ \sup_{r \in [0, T]} \|Y_r\|_V^2 \right] \|\zeta_n - \zeta\|_{\mathcal{C}^1([0, T]; D^*)}^2 T^2 \xrightarrow{n \rightarrow +\infty} 0, \end{aligned}$$

thus  $\int_0^t \langle Y_r, B^* \zeta_n(r) \rangle_V dB_r^H \xrightarrow{n \rightarrow +\infty} Y_t^2$  in  $L^2(\Omega)$ . By the closedness of Skorokhod integral,  $Y_t^2 = \int_0^t \langle Y_r, B^* \zeta(r) \rangle_V dB_r^H$  because

$$\mathbb{E} \left[ \int_0^t \langle Y_r, B^* (\zeta_n(r) - \zeta(r)) \rangle_V^2 dr \right] \leq \mathbb{E} \left[ \sup_{r \in [0, T]} \|Y_r\|_V^2 \right] \|B^*\|_{\mathcal{L}(V)}^2 \|\zeta_n - \zeta\|_{\mathcal{C}^1([0, T]; D^*)}^2 T \xrightarrow{n \rightarrow +\infty} 0$$

and  $\langle Y, B^* \zeta_n \rangle_V \in \text{Dom } \delta_H$  for any  $n \in \mathbb{N}$ .  $\square$

Now, we are able to prove the uniqueness result.

**Theorem 6.2.** *Under the assumptions of Theorem 4.6 the solution to the equation (6.1) is unique in the space  $\mathcal{M}$ .*

*Proof.* Let  $X^M$  be the solution to the equation

$$X_t^M = U_Y(t, 0)x + \int_0^t U_Y(t, r)F(r)dr, \quad t \in [0, T],$$

which is also a weak one to (6.1) (Theorem 4.6), where

$$U_Y(t, s) = S_B(B_t^H - B_s^H)U(t - s, 0), \quad s \leq t \leq T,$$

(for more details see (3.5)). Using the notation from Section 3

$$\|U_Y(t, s)\|_{\mathcal{L}(V)} \leq M_B \exp \{2\omega_B \|B^H\|_{\mathcal{C}([0, T])}\} C_U, \quad 0 \leq s \leq t \leq T,$$

it follows

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \|X_t^M\|_V^2 \right] &\leq 2C_U^2 M_B^2 \left( \|x\|_V^2 \mathbb{E} \exp \{2\omega_B \|B^H\|_{\mathcal{C}([0, T])}\} \right. \\ &\quad \left. + \|F\|_{L^2([0, T])}^2 T \mathbb{E} \exp \{4\omega_B \|B^H\|_{\mathcal{C}([0, T])}\} \right) < +\infty, \end{aligned}$$

by the Fernique Theorem. Therefore,  $X^M \in \mathcal{M}$  (the continuity of trajectories is guaranteed by Theorem 4.1).

Take another weak solution  $X^1 \in \mathcal{M}$  to (6.1) and define

$$\bar{X} := X^1 - X^M.$$

Then  $\bar{X}$  is a weak solution to the equation

$$d\bar{X}_t = A\bar{X}_t dt + B\bar{X}_t dB_t^H, \quad \bar{X}_0 = 0.$$

Hence, applying Lemma 6.1 to  $\langle \bar{X}_t, \xi \rangle_V$  for any fixed  $\xi \in D^*$  and  $\zeta(s) = S_A^*(t-s)\xi, s \in [0, t]$ , it follows

$$\begin{aligned} \langle \bar{X}_t, \xi \rangle_V &= \int_0^t \langle \bar{X}_r, A^* S_A^*(t-r)\xi - S_A^*(t-r)A^*\xi \rangle_V dr + \int_0^t \langle \bar{X}_r, B^* S_A^*(t-r)\xi \rangle_V dB_r^H \\ &= \int_0^t \langle S_A(t-r)B\bar{X}_r, \xi \rangle_V dB_r^H = \left\langle \int_0^t S_A(t-r)B\bar{X}_r dB_r^H, \xi \right\rangle_V. \end{aligned}$$

Thus Theorem 4.1 yields

$$\bar{X}_t = \int_0^t S_A(t-r)B\bar{X}_r dB_r^H = S_B(B_t^H)U(t, 0)0 = 0$$

and  $X^1 = X^M$ . □

**Corollary 6.3.** *The weak solution  $\{S_B(B_t^H)U(t, 0)x, t \in [0, T]\}$  to the equation (5.1) is unique in  $\mathcal{M}$ .*

*In particular, the solution*

$$X_t = \exp \left\{ bB_t^H - \frac{1}{2}b^2t^{2H} + at \right\} x, \quad t \in [0, T],$$

*to the one-dimensional equation*

$$dX_t = aX_t dt + bX_t dB_t^H, \quad X_0 = x,$$

*is unique in  $\mathcal{M}$ .*

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FACULTY OF MATHEMATICS AND PHYSICS, CHARLES UNIVERSITY IN PRAGUE, SOKOLOVSKÁ 83, 186 75 PRAGUE 8, CZECH REPUBLIC

E-mail address: maslow@karlin.mff.cuni.cz

DEPARTMENT OF MATHEMATICS, FACULTY OF CHEMICAL ENGINEERING, UNIVERSITY OF CHEMICAL TECHNOLOGY PRAGUE, STUDENTSKÁ 6, 166 28 PRAGUE 6 – DEJVICE, CZECH REPUBLIC

E-mail address: snuparkj@vscht.cz